GRID OPTIMIZATION FOR MATRIX-BASED SOURCE LOCALIZATION UNDER INHOMOGENEOUS SENSOR TOPOLOGY

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ABSTRACT

Herein, the problem of non-parametric source localization based on signal strength measured at different sensor locations is examined. A recently developed matrix-based method is considered. This method first arranges the measurements into an observation matrix based on a uniform grid defined in the target area and the sensor locations, and then exploits sparse matrix processing techniques to localize the source. This paper finds that the localization performance degrades when the spatial pattern of the sensors is highly non-uniform, and the uniform grid formation is only a suboptimal solution. Rather, the grid should be optimized according to the specific sensor topology. With the insight from the Cramér-Rao bound (CRB) analysis of matrix completion, a clustering problem is formulated to optimize the grid. It is demonstrated that with grid optimization, both the matrix completion and the source localization performance can be significantly improved. The proposed strategy is robust under inhomogeneous sensor topology and substantially outperforms weighted centroid localization (WCL) algorithms.

Index Terms— Grid optimization, matrix completion, source localization, clustering, unimodality

1. INTRODUCTION

This paper studies non-parametric source localization based on coarse sensor measurements, where there is no propagation model to exploit and the characteristics of the signal emitted from the source is unknown.

Such a problem is motivated by the demands of localizing a target from coarse measurement signals [1, 2]. For example, in wireless sensor networks, a malfunctioning node or a jammer may degrade the network performance, and the network may want to localize the node, but only side information is available, such as packet drop rate observed at each sensor node. In underwater localization, it is very difficult to determine the propagation characteristics, such as the sound speed and the path loss exponent, of acoustic signals in the time-varying water environment. In these scenarios, it is difficult to learn a parametric model for localization.

There exist some classical non-parametric source localization methods. First, it might be possible to first learn a model using data-driving approaches, and then jointly perform model-based localization, for example, using kernel regression and support vector machines [3–5]. Yet, these methods require much data and are sensitive to the choice of kernels. Second, much existing work focuses on WCL algorithms. The core idea is to estimate the source location as the weighted location of the sensors, where the weights can be heuristically designed as an increasing function of the strength of the signals received at each sensor [6–12]. However, it is non-trivial to pick an appropriate weighting scheme, and moreover, it is also known that WCL methods suffer from significant bias when the source locates away from the *centroid* of the sensor networks or when the sensors are spatially distributed in a highly *non-uniform* pattern [13, 14].

This paper tackles the issue of the *inhomogeneity* of the sensor topology for non-parametric source localization. In practice, the sensors are usually deployed in a non-uniform pattern. To address such inhomogeneity, we define non-uniform grid in the target area, and optimize the spacing of the grid according to the sensor topology. The measurements obtained at each sensor position are arranged into an observation matrix defined according to the non-uniform grid, and subsequently, the source location is estimated by matrix completion followed by peak localization of the dominant singular vectors of the completed observation matrix [15, 16]. This method only exploits the *structural* property that the signal strength decreases as the distances increases. Our results confirm that the proposed method is robust to the uncertain propagation environment and the inhomogeneity of the sensor topology.

Our contributions: First, we derive the CRB of matrix completion to analyze how the sensor topology and the grid formation would together affect the performance of the matrix completion, which is a critical step in matrix-based source

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localization. Second, we design a clustering problem to optimize the grid formation such that the matrix completion error is to be reduced. Finally, we perform numerical experiments to confirm that the matrix-based source localization under the optimized grid is indeed robust over various levels of inhomogeneity of sensor topologies, and the proposed strategy significantly outperform the WCL algorithm.

Relation to prior work: It is known that WCL methods suffer from large bias due to inhomogeneity of the sensor topology and source location being away from the center in [13]. Existing work developed mean-shift algorithms in [10] and a sensor selection approach [14], but their performance is sensitive to the heuristic choice of algorithm parameters. Our prior work developed matrix-based source localization methods [16, 17], but the methods require prescribed *uniform* grid to form the observation matrix and it was not known how to optimize the grid.

2. SYSTEM MODEL

2.1. The Non-parametric Localization Problem

Consider an active source located at $\boldsymbol{s} \in \mathbb{R}^2$. The signal emitted from the source is detected by M sensors with known locations $\boldsymbol{z}_m \in \mathbb{R}^2$, $m = 1, 2, \dots, M$, deployed randomly in an $L \times L$ area. The strength of the signal received by the mth source is given by

$$\gamma_m = f(d(\boldsymbol{s}, \boldsymbol{z}_m)) + \xi_m$$

where $d(\boldsymbol{s}, \boldsymbol{z}) = \|\boldsymbol{s} - \boldsymbol{z}\|_2$ describes the distance between the source at $\boldsymbol{s}$ and the sensor at $\boldsymbol{z}$, $f(d)$ describes the signal strength degradation in terms of distance d, and ξ_m is a random variable that captures the measurement noise. The problem is to estimate the source location $\boldsymbol{s}$ based on the measurements $\{(\boldsymbol{z}_m, \gamma_m)\}$ from the M sensors.

The challenge of such a source localization problem is that the propagation function $f(d)$ is unknown, except for a general property that $f(d)$ is a decreasing function of the distance d.

2.2. Matrix-based Source Localization

First, discretize the target area into N rows and N columns. Construct a matrix $\mathsf{H} \in \mathbb{R}^{N \times N}$, whose (i, j)th entry is given by $\mathsf{H}_{ij} = \gamma_m$ if the mth sensor locates in the (i, j)th grid.[1] The parameter N is chosen such that each row and each column have at least one measurement.

Second, the missing value of H can be found using matrix completion techniques. One possible way to complete H is to find the solution to the following convex optimization

[1] When there are more than one sensor locating in the same grid, then H_{ij} can be taken as the mean of these sensor measurements γ_m.

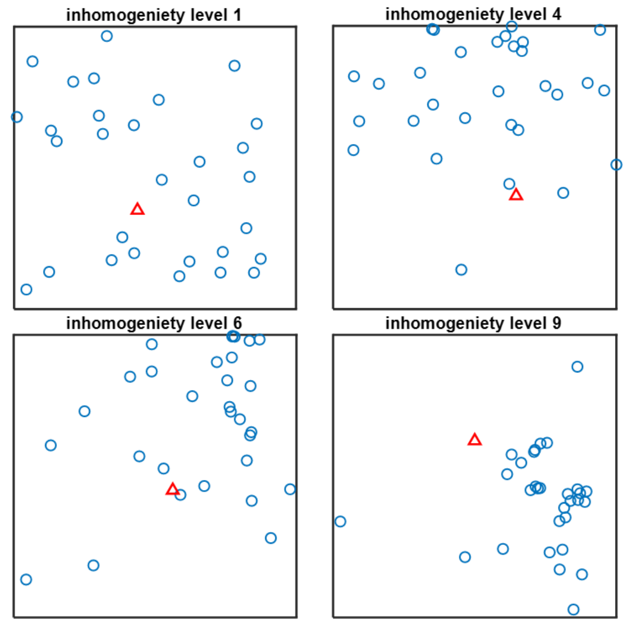

Fig. 1. Sensor topologies (blue circles) with different levels of inhomogeneity [18]. Uniform-grid-based or WCL algorithms would suffer from huge large bias for estimating the source (red triangle).

problem [16, 19]:

$$\begin{aligned} \underset{\boldsymbol{X} \in \mathbb{R}^{N \times N}}{\text{minimize}} \quad & \|\boldsymbol{X}\|_* \\ \text{subject to} \quad & X_{ij} = \mathsf{H}_{ij}, \qquad \forall (i, j) \in \Omega \end{aligned}$$

where $\|\boldsymbol{X}\|_*$ denotes the nuclear norm of $\boldsymbol{X}$ and Ω denotes the set of matrix entries (i, j) that are observed in H.

Then, it is known that the completed matrix $\bar{\mathsf{H}}$ is approximately unimodal. Here, a vector $\boldsymbol{u}$ is a unimodal vector if the entries $u_i \geq 0$ first increase and then decrease from $i = 1$ to N; moreover, a matrix $\boldsymbol{H}$ is unimodal if all its row and column vectors are unimodal. Consider a noise-free reference matrix $\boldsymbol{H}$ defined as $H_{ij} = f(d(\boldsymbol{s}, \boldsymbol{c}_{ij}))$, where $\boldsymbol{c}_{ij}$ is the center location of the (i, j)th grid. Then, $\bar{\mathsf{H}}$ is a noisy version of the reference matrix $\boldsymbol{H}$. It has been shown in [16] that $\boldsymbol{H}$ is unimodal, and in addition, the left and right dominant singular vectors are also unimodal with the peak locations being the source locations in the x-axis and y-axis, respectively.

As a result, the final step is to compute the dominant left and right singular vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively, of $\bar{\mathsf{H}}$, and to perform peak localization of the approximately unimodal vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Specifically, denote the x coordinates of the grid centers as $\boldsymbol{x}^{\text{c}} = (x_1^{\text{c}}, x_2^{\text{c}}, \dots, x_N^{\text{c}})$, and the y coordinates as $\boldsymbol{y}^{\text{c}} = (y_1^{\text{c}}, y_2^{\text{c}}, \dots, y_N^{\text{c}})$. Then, the peak localization of a vector $\boldsymbol{u}$ is to find a non-parametric method to first interpolate the N-point vector $\boldsymbol{u}$ along the coordinates $\boldsymbol{x}^{\text{c}}$ to obtain a continuous function $\tilde{u}(x)$ and then to find the maximizer $\hat{x} = \arg\max_x \tilde{u}(x)$. In this paper, we use an estimator presented in [16] that exploits the unimodality of $\boldsymbol{u}$ for peak localization. A similar method is applied to find $\hat{y}$ from the right dominant singular vector $\boldsymbol{v}$ and the y coordinates $\boldsymbol{y}^{\mathrm{c}}$ of the grid. The source location estimate is given by $\hat{\boldsymbol{s}} = (\hat{x}, \hat{y})$.

A critical unsolved issue in the above matrix-based localization is the formation of the grid. If, for a certain grid, there is too few observation in one of the row or column in H as shown in Fig. 1, then there could be large matrix completion error that eventually deteriorates the source localization performance. This paper proposes to optimize the grid positions, *i.e.*, $\boldsymbol{x}^{\mathrm{c}}$ and $\boldsymbol{y}^{\mathrm{c}}$ according to the sensor topology $\{\boldsymbol{z}_m\}$.

3. NON-UNIFORM GRID OPTIMIZATION FROM SENSOR TOPOLOGY

3.1. Cramér-Rao Bound Analysis for Matrix Completion

An intuitive guideline to optimize the grid coordinates $\boldsymbol{x}^{\mathrm{c}}$ and $\boldsymbol{y}^{\mathrm{c}}$ is to form a desirable observation pattern for the sparse matrix H such that a completed matrix $\bar{\mathsf{H}}$ could be obtained in a higher accuracy. Towards this end, we first analyze the CRB for unbiased low-rank matrix estimators $\bar{\mathsf{H}}(\boldsymbol{\gamma})$, where $\boldsymbol{\gamma} = \{\gamma_m\}$ is the sensor measurement data.

Consider $\boldsymbol{\gamma}$ to be distributed as $\mathcal{N}(\boldsymbol{A}\mathrm{vec}(\boldsymbol{H}), \boldsymbol{\Sigma})$, where $\boldsymbol{A} \in \mathbb{R}^{m\times N^2}$ is an indicator matrix with $A_{m,k} = 1$ if the mth sensor measurement is mapped to the kth entry in the vectorized observation matrix vec(H), *i.e.*, the mth sensor is mapped to the (i, j)th entry of H which corresponds to the kth entry of vec(H). In addition, $A_{m,k} = 0$, otherwise. It has been shown in [20] that $\mathbb{E}\{\|\boldsymbol{H} - \bar{\mathsf{H}}(\boldsymbol{\gamma})\|_{\mathrm{F}}^2\}$ is lower bounded by

$$\Gamma(\boldsymbol{H}) \triangleq \max\Big\{\mathrm{tr}\Big\{[(\boldsymbol{I}_N \otimes \boldsymbol{U}_0)^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{A}(\boldsymbol{I}_N \otimes \boldsymbol{U}_0)]^{-1}\Big\}, \\ \mathrm{tr}\Big\{[(\boldsymbol{V}_0 \otimes \boldsymbol{I}_N)^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{A}(\boldsymbol{V}_0 \otimes \boldsymbol{I}_N)]^{-1}\Big\}\Big\} \tag{1}$$

where $\boldsymbol{U}_0 = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r]$ is a matrix that contains the r dominant left singular vectors of the desired reference matrix $\boldsymbol{H}$ given rank$(\boldsymbol{H}) = r$, $\boldsymbol{V}_0 = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r]$ is the matrix for the dominant right singular vectors, and $\otimes$ denotes the Kronecker product.

Note that if the matrix $\boldsymbol{H}$ concentrates its energy mostly at its first dominant singular vector pair, then rank-1 approximation can be applied to $\boldsymbol{H}$ and $\boldsymbol{U}_0 \approx \boldsymbol{u}_1$, $\boldsymbol{V}_0 \approx \boldsymbol{v}_1$. The CRB in (1) can be approximated as the maximum between $\mathrm{tr}\{[(\boldsymbol{I}_N \otimes \boldsymbol{u}_1)^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{A}(\boldsymbol{I}_N \otimes \boldsymbol{u}_1)]^{-1}\}$ and $\mathrm{tr}\{[(\boldsymbol{v}_1 \otimes \boldsymbol{I}_N)^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{A}(\boldsymbol{v}_1 \otimes \boldsymbol{I}_N)]^{-1}\}$

Denote $m = M(i, j)$ as a mapping which indicates that the (i, j)th entry of H is obtained from the mth measurement γ_m. Denote the (m, m)th element of $\boldsymbol{\Sigma}$ as $\sigma^2_{M(i,j)}$. Since the sensor measurements γ_m and the deployment locations $\boldsymbol{z}_m$ are considered independently, the off-diagonal elements of $\boldsymbol{\Sigma}$ are considered to be zero. As a result, an equivalent experssion of (1) can be obtained as

$$\Gamma(\boldsymbol{H}) \approx \max\Big\{\sum_{i=1}^{N}\Big(\sum_{j:(i,j)\in\Omega}\frac{u_{1j}}{\sigma^2_{M(i,j)}}\Big)^{-1}, \\ \sum_{j=1}^{N}\Big(\sum_{i:(i,j)\in\Omega}\frac{v_{1i}}{\sigma^2_{M(i,j)}}\Big)^{-1}\Big\} \tag{2}$$

where u_{1j} is the jth element of the dominant singular vector $\boldsymbol{u}_1$ and v_{1i} is the ith element of $\boldsymbol{v}_1$. The above expression is computed in a straight-forward way from (1) using rank-1 approximations and by associating the indicator matrix $\boldsymbol{A}$ with the observation set Ω. The detail steps are omitted here due to the page limit.

Although it is difficult to directly optimize (2) as $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ are not available, the matrix structure observed from (2) can be exploited. First, it is required to have at least one observation in each row or column, and otherwise, either term in (2) will become too large. Such a property provides some guideline on the choice of the matrix dimension N. Second, it is also desired to reduce the noise term $\sigma^2_{M(i,j)}$, which comes from two factors: (i) the measurement noise $\xi_{M(i,j)}$, and (ii) the discretization noise $f(d(\boldsymbol{s}, \boldsymbol{z}_{M(i,j)})) - H_{ij}$ due to not measuring at the grid center $\boldsymbol{c}_{ij} = (x_i^{\mathrm{c}}, y_j^{\mathrm{c}})$, where $H_{ij} = f(d(\boldsymbol{s}, \boldsymbol{c}_{ij}))$ and recall that $\mathsf{H}_{ij} = \gamma_{M(i,j)} = f(d(\boldsymbol{s}, \boldsymbol{z}_{M(i,j)})) + \xi_{M(i,j)}$. Note that, since $f(d)$ is a decreasing function of d, reducing the distance $\|\boldsymbol{z}_{M(i,j)} - \boldsymbol{c}_{ij}\|_2$ reduces the noise variance $\sigma^2_{M(i,j)}$. Such an observation motivates our grid optimization that minimizes the distance from the sensor locations to the corresponding grid centers.

3.2. Grid Optimization

Recall that $M(i, j; \boldsymbol{x}^{\mathrm{c}}, \boldsymbol{y}^{\mathrm{c}})$ maps the mth sensor measurement at $\boldsymbol{z}_m$ to the (i, j)th entry of H according to the grid formation defined by the x and y coordinates $\boldsymbol{x}^{\mathrm{c}}$ and $\boldsymbol{y}^{\mathrm{c}}$, respectively. Using $M(i, j)$ for brevity and aiming at a minimum total sensor-to-grid-center distance, the grid optimization problem can be formulated as follows

$$\underset{\{x_i^{\mathrm{c}}\},\{y_j^{\mathrm{c}}\}}{\text{minimize}} \quad \sum_{i,j}\sum_{m\in M(i,j)}\|\boldsymbol{z}_m - (x_i^{\mathrm{c}}, y_j^{\mathrm{c}})\|. \tag{3}$$

Solving problem (3) is generally difficult, but the problem can be decomposed into an x-subproblem and a y-subproblem if L_1-norm distance is considered:

$$\underset{\{x_i^{\mathrm{c}}\}}{\text{minimize}} \quad \sum_{i=1}^{N}\sum_{m\in\mathcal{R}_i}|z_{m,1} - x_i^{\mathrm{c}}| \tag{4}$$

$$\underset{\{y_j^{\mathrm{c}}\}}{\text{minimize}} \quad \sum_{i=1}^{N}\sum_{m\in\mathcal{C}_i}|z_{m,2} - y_j^{\mathrm{c}}| \tag{5}$$

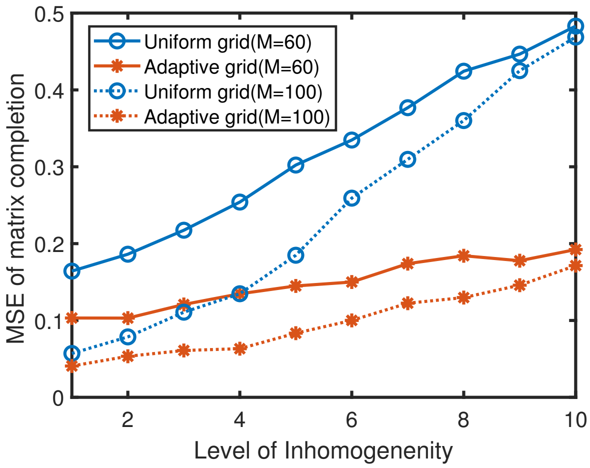

Fig. 2. Matrix completion error substantially decreases under the adaptive grid formation.

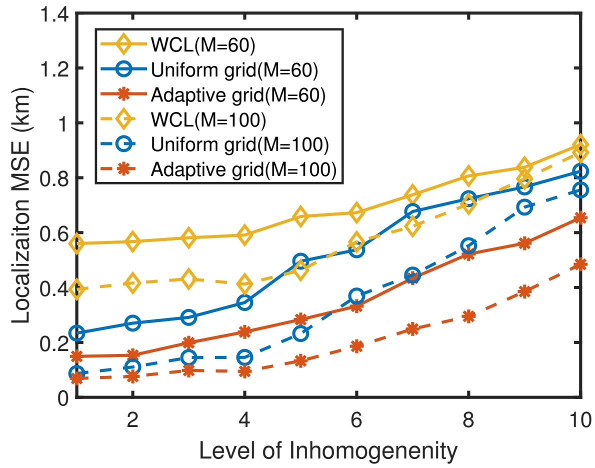

Fig. 3. Localization error under various sensor topology.

where $(z_{m,1}, z_{m,2})$ represents the xy coordinates of $\boldsymbol{z}_m$, $\mathcal{R}_i = \{m : |z_{m,1} - x_i^{\mathrm{c}}| < |z_{m,1} - x_k^{\mathrm{c}}|, \forall k \neq i\}$ is the subset of sensors that should be assigned to the ith row according to the shortest L_1-distance rule, and $\mathcal{C}_j = \{m : |z_{m,2} - y_j^{\mathrm{c}}| < |z_{m,2} - y_k^{\mathrm{c}}|, \forall k \neq j\}$ is the subset of sensors that should be assigned to the jth column according to the shortest L_1-distance.

Problem (4)–(5) essentially resembles two one-dimensional clustering problems based on the L_1 metric. A widely known algorithm to solve the clustering problems (4)–(5) is the K-means algorithm, which iteratively performs the following two steps until convergence:

- Assignment step: Given row coordinates $\{x_i^{\mathrm{c}}\}$, assign each sensor m to the corresponding set $\mathcal{R}_i$, if the x-coordinate $z_{m,1}$ of $\boldsymbol{z}_m$ satisfies $|z_{m,1} - x_i^{\mathrm{c}}| < |z_{m,1} - x_k^{\mathrm{c}}|, \forall k \neq i$. Compute the column assignment $\mathcal{C}_j$ based on the y-coordinate $z_{m,2}$ in a similar way.
- Averaging step: Based on the assignments $\mathcal{R}_i$, update the row coordinates of the grid as $x_i^{\mathrm{c}} = \frac{1}{|\mathcal{R}_i|} \sum_{m \in \mathcal{R}_i} z_{m,1}$. Update the column coordinates y_j^{c} based on $\mathcal{C}_j$ in a similar way.

It is well-known that the K-means algorithm always converges and the iterate monotonically decreases the objectives (4)–(5). As a result, if one uses the uniform grid formation to initialize K-means, then the K-means solution must generate a grid formation with total sensor-to-grid-center distance no larger than the uniform formation.

4. NUMERICAL RESULTS

We consider localizing an active underwater source in an $L \times L$ area where $L = 2$ kilometers. The sensors collect the energy emitting from the source and the measurement is simulated as $\gamma = (1 + d^{1.5} A(f)^d)^{-1} + \xi$ where $10\log_{10} A(f) = 0.11 f^2/(1+f^2) + 44 f^2/(4100+f^2) + 2.75 \times 10^{-4} f^2 + 0.003$ where $f = 5$ kHz, d is the distance from source to the location of the sensor, and $\xi \sim \mathcal{N}(0, \sigma^2)$ is to model the noise with $\sigma = 3$ dB.

The simulation is performed for adaptive grid method (our proposed method), uniform grid method and WCL method which serves as the baseline. In the WCL method, $\hat{\boldsymbol{s}}_{\mathrm{WCL}} = \sum_{m=1}^{M} w_m \boldsymbol{z}_m / \sum_{m=1}^{M} w_m$ is used to estimate the location of the source, where $w_m = \gamma_m$ serves as the weight.

Fig. 2 compares the matrix completion error of our proposed method to uniform grid method for $M = \{60, 100\}$ sensors under different level of inhomogeneity. The results demonstrate that the proposed method outperforms the uniform method under ALL levels of inhomogeneity.

Fig. 3 shows the localization error with respect to different inhomogeneity levels of sensor topology and each sensor topology contains $M = \{60, 100\}$ sensors. The dimension of the constructed grid was chosen to be $N = 15$. Our proposed method demonstrates an improvement in the localization accuracy compared to the uniform grid method and the WCL method under ALL levels of inhomogeneity.

5. CONCLUSIONS

In this paper, we proposed a grid optimization method for source localization with the knowledge of sensor deployment. Based on our derivation of CRB for the matrix completion error, an adaptive grid formation method was developed. Numerical results show that our proposed method substantially improves the localizaiton accuracy under different inhomogeneity levels of sensor topologies and the proposed method significantly outperforms WCL schemes.

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