REGRESSION ASSISTED MATRIX COMPLETION FOR RECONSTRUCTING A PROPAGATION FIELD WITH APPLICATION TO SOURCE LOCALIZATION

Hao Sun and Junting Chen

School of Science and Engineering (SSE) and Future Network of Intelligence Institute (FNii)
The Chinese University of Hong Kong, Shenzhen, Guangdong 518172, China

ABSTRACT

This paper develops a regression assisted matrix completion method to reconstruct the propagation field for received signal strength (RSS) based source localization without prior knowledge of the propagation model. Existing matrix completion methods did not exploit the fact that the uncertainty of each observed entry is different due to the reality that the sensor density may vary across different locations. This paper proposes to employ local polynomial regression to increase the accuracy of matrix completion. First, the values of selected entries of a matrix are estimated via interpolation from local measurements, and the interpolation error is analyzed. Then, a matrix completion problem that is aware of the different uncertainty of observed entries is formulated and solved. It is demonstrated that the proposed method significantly improves the performance of matrix completion, and as a result, increases the localization accuracy from the numerical results.

***Index Terms*—** Propagation field reconstruction, source localization, local polynomial regression, matrix completion, uncertainty

1. INTRODUCTION

RSS based active source localization is an essential problem with many applications [1, 2]. Existing approaches are usually based on the assumption or estimation of a propagation field. For example, [3, 4] assumed that the source energy decays proportional to the inverse of the squared distance. Thus, the source location can be found via exploiting the estimated propagation model.

However, it is generally difficult to characterize a model for a signal propagation field. For radio signals in an urban environment, the urban structure shapes signal reflection and diffraction, creating an environment-dependent propagation field. For acoustic signals in underwater communication, the energy decay law varies significantly with respect to several factors such as temperature, ocean current, and water salinity. Some existing approaches for based source localization under harsh environments employ machine learning methods to construct the propagation field implicitly. For example, [5] and [6] respectively used kernel regression and support vector machine for source localization, where the kernel functions may serve as a proxy of the propagation field. However, these methods may either require a large amount of data or be sensitive to the choice of kernel functions. Other approaches use weighted centroid localization (WCL) [7–9] techniques to estimate the source localization as a weighted sum of the sensor locations where the weights depend on the sensor measurements. However, this method may suffer from a large bias due to the non-uniform distribution of the sensors with respect to the source location.

This paper focuses on reconstructing the propagation field with the aim of source localization in harsh environment with no prior knowledge of the propagation law. Related methods can be mainly put into two categories. Classical methods include Kriging [10, 11] and local polynomial regression [12] to interpolate the entire propagation field from a number of samples. While these interpolation-type methods perform well in the region with dense samples, they can perform badly in a local area with sparse samples. These methods do not exploit the structural property of a propagation field, where the closer to the source, the stronger the signal, and they are not suitable for source localization as the ultimate goal. The other category of methods are based on matrix completion. They form a sparse matrix based on the sensor measurements at different locations, and the missing values are found by solving a sparse matrix completion problem assuming that the matrix is low rank due to the structural property of the propagation [13, 14]. However, it may not be straightforward to perform matrix completion. The alternating least square method (ALS) [15, 16] and the fixed point continuation with approximate SVD (FPCA) method [17] require the knowledge of the matrix rank, and [18, 19] are not aware of measurement noise for each specific entry when performing nuclear norm minimization method.

In this paper, we study noise aware matrix completion by exploiting local polynomial regression for the formation of

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the sparse matrix. Specifically, for a local area with dense measurements, an observed entry of a sparse matrix is formed via polynomial regression; on the other hand, for a local area with sparse measurement, it yields a missing value. With such a strategy, we formulate a new matrix completion problem that is aware of the different uncertainty on each observed entries. Our results confirm that the proposed regression assisted matrix completion approach achieves smaller mean squared error (MSE) in matrix completion error, which implies a better reconstruction of the signal propagation field, leading to a higher accuracy in non-parametric source localization.

2. SYSTEM MODEL

2.1. Source Localization via Matrix Formation

Consider an active source located at $\boldsymbol{s} \in \mathcal{D} \subset \mathbb{R}^2$. The signal emitted from the source is detected by M sensors with known locations $\boldsymbol{z}_m \in \mathbb{R}^2$, $m = 1, 2, \ldots, M$, randomly deployed in an $L \times L$ area $\mathcal{D}$. The strength of the signal received by the mth source is given by

$$\gamma_m = g(d(\boldsymbol{s}, \boldsymbol{z}_m)) + \epsilon_m$$

where $d(\boldsymbol{s}, \boldsymbol{z}) = \|\boldsymbol{s} - \boldsymbol{z}\|_2$ describes the distance between the source at $\boldsymbol{s}$ and the sensor at $\boldsymbol{z}$, $g(d)$ describes the RSS in terms of the propagation distance d, and ϵ_m is a random variable with zero mean and variance σ^2.

However, the propagation model $g(d)$ is unknown, except that $g(d)$ is believed to be decreasing in distance d. Therefore, we propose to estimate the source location $\boldsymbol{s}$ by first reconstructing the propagation field $g(d(\boldsymbol{s}, \boldsymbol{z}_m))$ using matrix completion methods; then, the source location can be estimated by the peak localization of the matrix. The general procedure can be described as follows [13, 14].

Discretize the target area $\mathcal{D}$ into N rows and N columns which results in N^2 grid cells in total. Construct a matrix $\hat{\boldsymbol{H}}$ with the entries representing the estimates of the signal strength measured at the corresponding grid cells. Specifically, let $\boldsymbol{c}_{ij}$ be the center location of the (i,j)th grid cell. Then, $\hat{H}_{ij}$ is constructed as the estimation of the received signal strength $g(d(\boldsymbol{s}, \boldsymbol{c}_{ij}))$ at location $\boldsymbol{c}_{ij}$. In prior studies [18, 20], it has been found that if the estimations are perfect, *i.e.*, $\hat{\boldsymbol{H}}$ is a complete matrix and its entries satisfy $\hat{H}_{ij} = g(d(\boldsymbol{s}, \boldsymbol{c}_{ij}))$, the matrix $\hat{\boldsymbol{H}}$, which represents the discretized propagation field $g(d(\boldsymbol{s}, \boldsymbol{z}))$, is likely low rank; moreover, the dominant singular vectors $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ of $\hat{\boldsymbol{H}}$ are unimodal, and the peak locations of $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ respectively represent the x-position and y-position of the source $\boldsymbol{s}$. Here, a vector $\boldsymbol{u} = (u_1, u_2, ..., u_N)$ is unimodal if its entries first increase and then decrease, $u_1 \leq u_2 \leq \cdots \leq u_{n_0}$, $u_{n_0} \geq u_{n_0+1} \geq \cdots \geq u_N$, for some $1 \leq n_0 \leq N$.

2.2. Matrix Completion under Uncertainty

However, it is challenging to form a complete observation matrix $\hat{\boldsymbol{H}}$ since there are just M measurements γ_m from the M randomly scattered sensors. Even for $M \geq N^2$, there may still be unobserved entries for $\hat{H}_{ij}$ due to the randomness of the sensor location. Classical methods may perform interpolation for $\hat{H}_{ij}$ based on a subset of sensors in the neighborhood of $\boldsymbol{c}_{ij}$. However, interpolation may not work if there are too few sensors nearby, and it can be very difficult to choose an appropriate model for the interpolation since the parametric form of $g(d)$ is unknown. Recent advance in sparse matrix completion enables a solution that first forms a sparse matrix from the grid cells that have sensors, and then fills in the missing value $\hat{H}_{ij}$ for the grid cells with no sensors. Yet, matrix completion may not work if there are too few observations in a row or a column.

This paper proposes to form a complete matrix $\bar{\boldsymbol{H}}$ using regression assisted matrix completion and tries to combine the advantages of local polynomial regression and matrix completion. First, if there are several sensors located within a radius b from a grid center $\boldsymbol{c}_{ij}$, then a regression method is used to estimate $\hat{H}_{ij}$ based on γ_m from these nearby sensors as illustrated in Fig. 1b. This forms a sparse matrix $\hat{\boldsymbol{H}}$. Second, a matrix $\bar{\boldsymbol{H}}$ is completed from $\hat{\boldsymbol{H}}$ by solving the following optimization problem [21]:

$$\begin{aligned} \underset{\boldsymbol{X} \in \mathbb{R}^{N \times N}}{\text{minimize}} \quad & \|\boldsymbol{X}\|_* \\ \text{subject to} \quad & |X_{ij} - \hat{H}_{ij}| \leq \bar{\epsilon}_{ij}, \qquad \forall (i,j) \in \Omega \end{aligned} \tag{1}$$

where $\|\boldsymbol{X}\|_*$ represents the nuclear norm of $\boldsymbol{X}$ and $\bar{\epsilon}_{ij}$ represents the uncertainty of $\hat{H}_{ij}$ for the (i,j)th grid, and Ω represents the index set of the observed entries in $\hat{\boldsymbol{H}}$.

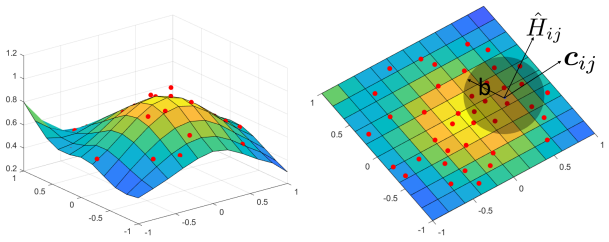
While there are a number of formulations for matrix completion in the literature, the above formulation (1) has the following advantages. First, it does not require the knowledge of the matrix rank as the ALS method [15, 16] does. Second, it tries to exploit the knowledge that the observation $\hat{H}_{ij}$ may contain uncertainty up to $\bar{\epsilon}_{ij}$.

As a result, it is crucial to choose the right parameter $\bar{\epsilon}_{ij}$ as it significantly affects the matrix completion performance. Prior work [18, 19] simply chooses a universal parameter $\bar{\epsilon}$ for the matrix completion problem (1), which is certainly *not* the optimal way. This paper provides a systematic method to estimate $\hat{H}_{ij}$ and determine an upper bound $\bar{\epsilon}_{ij}$ of the level of uncertainty. Fig. 1 shows the local polynomial regression method used in estimating the matrix values.

3. EXPLOITING LOCAL POLYNOMIAL REGRESSION TO ASSIST MATRIX COMPLETION

3.1. A Local Polynomial Regression Approach for $\hat{H}_{ij}$

For a fixed source location $\boldsymbol{s}$, define $\rho(\boldsymbol{z}) = g(d(\boldsymbol{s}, \boldsymbol{z}))$ for notation simplicity. Consider to approximate $\rho(\boldsymbol{z})$ in the neigh-



(a) (b)

Fig. 1: (a) Propagation field reconstruction based on local polynomial regression (b) Sensor measurements (red dots) within a range of b (grey region) from the grid center $\boldsymbol{c}_{ij}$ are used to estimate $\hat{H}_{ij}$.

borhood of $\boldsymbol{c}$ using a parametric model $\hat{\rho}(\boldsymbol{z};\boldsymbol{c})$. Here, we focus on small order polynomial models, such as the zero-th order model

$$\hat{\rho}(\boldsymbol{z};\boldsymbol{c}) = \alpha(\boldsymbol{c}) \tag{2}$$

and the first order model

$$\hat{\rho}(\boldsymbol{z};\boldsymbol{c}) = \alpha(\boldsymbol{c}) + \boldsymbol{\beta}^{\mathrm{T}}(\boldsymbol{c})(\boldsymbol{z}-\boldsymbol{c}) \tag{3}$$

where the coefficients $\alpha(\boldsymbol{c})$ and $\boldsymbol{\beta}(\boldsymbol{c})$ depend on the approximation center $\boldsymbol{c}$.

The coefficients of the local model $\hat{\rho}(\boldsymbol{z};\boldsymbol{c})$ are computed based on the nearby observations γ_m. It is clear that the observation that is closer to the approximation center $\boldsymbol{c}$ should have a larger weight. Therefore, we impose a weight for γ_m based on the distance $||\boldsymbol{z}_m - \boldsymbol{c}||$ as $w_m(\boldsymbol{c}) = K(\frac{||\boldsymbol{z}_m-\boldsymbol{c}||}{b})$, where $K(u)$ is a kernel function which is, optionally, chosen as the Epanechnikov function here, and b is the radius of the observation window [22].

We adopt a least-squares regression approach and the coefficients of the local model $\hat{\rho}(\boldsymbol{z};\boldsymbol{c})$ are determined as the solution to the following distance-weighted regression problem:

$$\underset{\boldsymbol{\theta}}{\text{minimize}} \sum_{m=1}^{M} w_m(\boldsymbol{c})\big(\gamma_m - \hat{\rho}(\boldsymbol{z}_m;\boldsymbol{c})\big)^2 \tag{4}$$

where $\boldsymbol{\theta} = \{\alpha(\boldsymbol{c}), \boldsymbol{\beta}(\boldsymbol{c}), ...\}$ is set of coefficients of the local polynomial $\hat{\rho}(\boldsymbol{z};\boldsymbol{c})$.

Proposition 1. *The solution to* (4) *under the zero-th order model* (2) *is given by*

$$\hat{\alpha}(\boldsymbol{c}) = \frac{\sum_{m=1}^{M} w_m(\boldsymbol{c})\gamma_m}{\sum_{m=1}^{M} w_m(\boldsymbol{c})}. \tag{5}$$

Proposition 2. *The solution to* (4) *under the first order model* (3) *is given by*

$$\begin{bmatrix} \hat{\alpha}(\boldsymbol{c}) \\ \hat{\boldsymbol{\beta}}(\boldsymbol{c}) \end{bmatrix} = (\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{Z})^{-1}\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{\gamma} \tag{6}$$

where

$$\boldsymbol{Z} = \begin{bmatrix} 1 & (\boldsymbol{z}_1-\boldsymbol{c})^T \\ \vdots & \vdots \\ 1 & (\boldsymbol{z}_M-\boldsymbol{c})^T \end{bmatrix}, \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_M \end{bmatrix}, \boldsymbol{W} = diag\{w_m(\boldsymbol{c})\}.$$

The zero-th order estimator $\hat{\alpha}(\boldsymbol{c})$ in (5) resembles the inverse distance weighting interpolation for $\rho(\boldsymbol{c})$ where the first order estimator (6) is used to analyze the estimation error in section 3.2. Therefore, we set $\hat{H}_{ij} = \hat{\alpha}(\boldsymbol{c}_{ij})$. Then, a sparse matrix $\hat{\boldsymbol{H}}$ is constructed.

3.2. Characterization of the Uncertainty $\bar{\epsilon}_{ij}$

We analyze the estimation error for $\hat{H}_{ij}$ under the regression solution (5) to the zero-th order model.

Define $\xi_{ij} \triangleq \hat{\alpha}(\boldsymbol{c}_{ij}) - \rho(\boldsymbol{c}_{ij})$, where $\hat{\alpha}$ is given in (5). The bias $\mathbb{E}\{\xi_{ij}\}$ and the variance $\mathbb{V}\{\xi_{ij}\}$ can be derived and summarized in the following theorem.

Theorem 1. *The bias and variance of estimation error* ξ_{ij} *under zero-th order local polynomial regression are*

$$\mathbb{E}\{\xi_{ij}\} = \frac{\nabla\rho(\boldsymbol{c}_{ij})\sum_{m=1}^{M}(\boldsymbol{z}_m-\boldsymbol{c}_{ij})\,w_m(\boldsymbol{c}_{ij})}{\sum_{m=1}^{M} w_m(\boldsymbol{c}_{ij})} + o(b)$$

$$\mathbb{V}\{\xi_{ij}\} = \frac{\sum_{m=1}^{M} w_m^2(\boldsymbol{c}_{ij})\sigma^2}{\sum_{m=1}^{M} w_m(\boldsymbol{c}_{ij})\sum_{m=1}^{M} w_m(\boldsymbol{c}_{ij})}$$

where $\nabla\rho(\boldsymbol{c}_{ij}) = [\frac{d\rho(\boldsymbol{c}_{ij})}{dx}\ \frac{d\rho(\boldsymbol{c}_{ij})}{dy}]$, and $o(b)$ represents a term that scales to zero faster than b as b goes to zero.

Proof. (sketch) The result can be derived by applying first-order Taylor's expansion to $\rho(\boldsymbol{z})$ at the neighborhood of $\boldsymbol{c}_{ij}$ and noticing that the residual from the Taylor's expansion scales as $o(||\boldsymbol{z}_m - \boldsymbol{c}_{ij}||)$ with a quantity $||\boldsymbol{z}_m - \boldsymbol{c}_{ij}|| < b$ due to our sampling strategy. □

Remark. The $\nabla\rho(\boldsymbol{c}_{ij})$ which represents the slope of the propagation field can be approximated using the first order model solution $\hat{\boldsymbol{\beta}}^{\mathrm{T}}(\boldsymbol{c}_{ij})$ estimated in (6). Then, the bias is proportional to the slope of the propagation field. In addition, the variance is proportional to σ^2 and the coefficients becomes $\frac{\sum_{m=1}^{M} w_m^2(\boldsymbol{c}_{ij})}{\left(\sum_{m=1}^{M} w_m(\boldsymbol{c}_{ij})\right)^2}$, when $w_1 = w_2 \cdots = w_M$, in which case, it attains the smallest value.

In the following, we use a $1-\delta$ confidence interval to construct the uncertainty level $\bar{\epsilon}_{ij}$ from the bias and the variance of the estimation $\hat{H}_{ij}$.

Define $\mu_{ij} = \frac{\hat{\boldsymbol{\beta}}^{\mathrm{T}}(\boldsymbol{c}_{ij})\sum_{m=1}^{M}(\boldsymbol{z}_m-\boldsymbol{c}_{ij})w_m(\boldsymbol{c}_{ij})}{\sum_{m=1}^{M} w_m(\boldsymbol{c}_{ij})}$ and $\nu_{ij}^2 = \frac{\sum_{m=1}^{M} w_m^2(\boldsymbol{c}_{ij})\sigma^2}{\sum_{m=1}^{M} w_m(\boldsymbol{c}_{ij})\sum_{m=1}^{M} w_m(\boldsymbol{c}_{ij})}$ to represent an estimation of the bias $\mathbb{E}\{\xi_{ij}\}$ and variance $\mathbb{V}\{\xi_{ij}\}$ in Theorem 1.

We roughly approximate all possible values for ξ_{ij} as in Gaussian distribution $\mathcal{N}(\mu_{ij}, \nu_{ij}^2)$. Construct a confidence

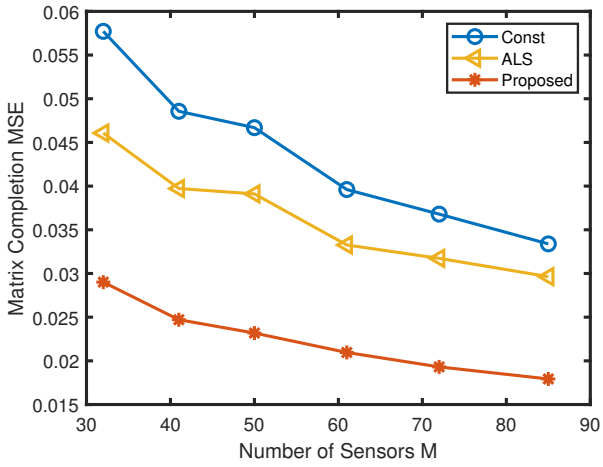


Fig. 2: MSE of Matrix Completion Versus Different Number of Sensors

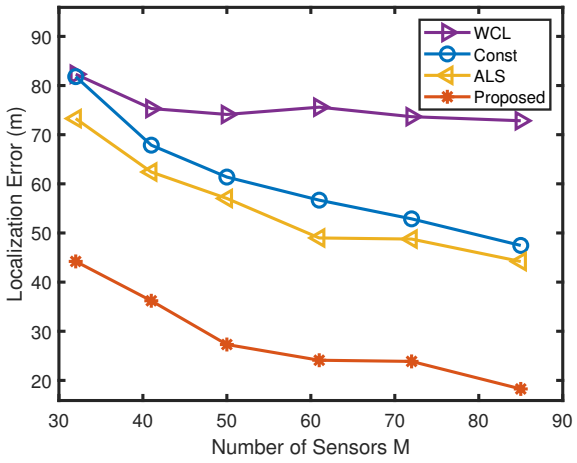


Fig. 3: Localization Error Versus Different Number of Sensors

interval for the ξ_{ij} of the probability $1-\delta$ which yields $P(-\eta_\delta \leq \frac{\xi_{ij}-\mu_{ij}}{\nu_{ij}} \leq \eta_\delta) = 1-\delta$ where $\eta_\delta = \Phi^{-1}(1-\frac{\delta}{2})$, and Φ is the cumulative distribution function (CDF) of the standard normal distribution. Therefore, the $1-\delta$ confidence interval of ξ_{ij} is $(\mu_{ij}-\eta_\delta\nu_{ij},\ \mu_{ij}+\eta_\delta\nu_{ij})$. As a result, we propose to choose the uncertainty parameter as $\bar{\epsilon}_{ij} = \max(|\mu_{ij}-\eta_\delta\nu_{ij}|,\ |\mu_{ij}+\eta_\delta\nu_{ij}|)$.

Up to now, the matrix value $\hat{H}_{ij}$ is estimated and the uncertainty $\bar{\epsilon}_{ij}$ of $\hat{H}_{ij}$ is constructed. Through solving (1), a complete matrix $\hat{\boldsymbol{H}}$ can be generated.

4. NUMERICAL RESULTS

Considering that there is an $L \times L$ underwater area with $L = 2$ kilometers. We deploy M sensors uniformly at random to measure the RSS emitting from an unknown signal source and we model the measurements as $\gamma = (1+d^{1.5}A(f)^d)^{-1} + \zeta$ where $10\log_{10}A(f) = 0.11f^2/(1+f^2) + 44f^2/(4100+f^2) + 2.75\times10^{-4}f^2 + 0.003$ [23] where $f = 5$ kHz, d is the distance between the sensor location and signal source, and ζ is a Gaussian noise with zero mean and standard variance $\sigma = 0.01$.

The simulation is performed for regression assisted method (proposed), constant uncertainty method [18], ALS method [15] and WCL method. In the regression assisted method, we choose a 95% confidence level with $\delta = 0.05$. According to (online) cross-validation through minimizing the MSE $\rho(\boldsymbol{c}_{ij}) - \hat{\rho}(\boldsymbol{z};\boldsymbol{c}_{ij})$ for the polynomial regression, the window parameter b was set as $b = 0.52$ (km). In the constant uncertainty method, we give the uncertainty $\bar{\epsilon}_{ij}$ a global value, e.g, we set all $\bar{\epsilon}_{ij} = 0.01$. In the ALS method, a matrix $\boldsymbol{X}$ is completed through alternatingly minimize $||\boldsymbol{y} - \boldsymbol{A}\text{vec}(\boldsymbol{X})||$ with $\boldsymbol{X} = \boldsymbol{LR}$, we set the rank of $\boldsymbol{X}$ matrix to be 1 as we found that the ALS baseline performs the best under the rank-1 model. The WCL method computes $\hat{\boldsymbol{s}}_{\text{WCL}} = \sum_{m=1}^{M} w_m \boldsymbol{z}_m / \sum_{m=1}^{M} w_m$ to estimate the location of the source, where $w_m = \gamma_m$ serves as the weight.

Fig. 1 compares the matrix completion error with respect to different number of sensors M where $M = 0.5 \times N^2$ with matrix dimension N varying from 8 to 13. We choose the integer part of M and the parameter 0.5 was chosen according to a cross-validation approach for a good performance for all schemes. The results demonstrate that the proposed method outperforms the constant uncertainty method and ALS method regardless of the number of sensors and dimension of matrix with a significant improvement on matrix completion accuracy.

Fig. 2 shows the localization error with respect to different number of sensors with the same setting as in Fig. 1. The results show that our proposed method significantly outperforms the constant uncertainty method, ALS method, and WCL method, which proves that the local polynomial regression to estimate $\hat{H}_{ij}$ and uncertainty construction for each $\hat{H}_{ij}$ really helps with the nuclear norm matrix completion algorithm.

5. CONCLUSION

In this paper, a regression assisted matrix completion method is proposed to construct a propagation field for non-parametric source localization. We use a local polynomial regression approach to estimate the matrix value and perform a bias and variance analysis to estimate the estimation error which is used to construct the uncertainty of matrix value. The simulation results show that our proposed method improves the matrix completion accuracy which leads to a better localization performance.

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